

In the name of Allah, the Beneficent, the Merciful

A radius of absolute convergence for power series in many variables

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Abstract

In this paper for power series in many (real or complex) variables a radius of (absolute) convergence is offered. This radius can be evaluated by a formula similar to Cauchy-Hadamard formula and in one variable case they are same.

Mathematics Subject Classification: 32A05, 15A60.

Key words: multiindex, power series in many variables.

In [1] a matrix representation of polynomial maps was offered and by the use of a new product of matrices the matrix representation of composition of polynomial maps was given. In this paper an application of this product to power series in many (real or complex) variables is presented. Namely, by the use of this product power series in many variables are presented in the form of power series in one variable. Then by the use of ρ -norm (defined in [1]) and the Cauchy-Hadamard type formula a radius of (absolute) convergence for such series is introduced and investigated. In one variable case it is the same Cauchy-Hadamard formula for radius of convergence of power series in one variable.

Here are some definitions and results related to the new product introduced in [1].

For a positive integer n let I_n stand for all row n -tuples with nonnegative integer entries with the following linear order: $\beta = (\beta_1, \beta_2, \dots, \beta_n) < \alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ if and only if $|\beta| < |\alpha|$ or $|\beta| = |\alpha|$ and $\beta_1 > \alpha_1$ or $|\beta| = |\alpha|$, $\beta_1 = \alpha_1$ and $\beta_2 > \alpha_2$ etcetera, where $|\alpha|$ stands for $\alpha_1 + \alpha_2 + \dots + \alpha_n$.

It is clear that for $\alpha, \beta, \gamma \in I_n$ one has $\alpha < \beta$ if and only if $\alpha + \gamma < \beta + \gamma$. We write $\beta \ll \alpha$ if $\beta_i \leq \alpha_i$ for all $i = 1, 2, \dots, n$, $\binom{\alpha}{\beta}$ stands for $\frac{\alpha!}{\beta!(\alpha-\beta)!}$, $\alpha! = \alpha_1! \alpha_2! \dots \alpha_n!$.

In future n and n' are assumed to be any fixed positive integers. Let F stand for the field of real or complex numbers.

For any nonnegative integer numbers p, p' let $M_{n,n'}(p, p'; F) = M(p, p'; F)$ stand for all " $p \times p'$ " size matrices $A = (A_{\alpha, \alpha'})_{|\alpha|=p, |\alpha'|=p'}$ (α presents row, α' presents column and $\alpha \in I_n, \alpha' \in I_{n'}$) with entries from F . Over such kind matrices in addition to the ordinary sum and product of matrices we consider the following "product" \odot as well:

Definition 1. If $A \in M(p, p'; F)$ and $B \in M(q, q'; F)$ then $A \odot B = C \in M(p+q, p'+q'; F)$ that for any $|\alpha| = p+q$, $|\alpha'| = p' + q'$, where $\alpha \in I_n, \alpha' \in I_{n'}$,

$$C_{\alpha, \alpha'} = \sum_{\beta, \beta'} \binom{\alpha}{\beta} A_{\beta, \beta'} B_{\alpha-\beta, \alpha'-\beta'}$$

, where the sum is taken over all $\beta \in I_n, \beta' \in I_{n'}$, for which $|\beta| = p, |\beta'| = p', \beta \ll \alpha$ and $\beta' \ll \alpha'$.

Let us agree that h (H, v, V) stands for any element of $M(0, 1; F)$ (respect. $M(0, p; F)$, $M(1, 0; F)$, $M(p, 0; F)$), where p may be any nonnegative integer). We use E_k for " $k \times k$ " size ordinary unit matrix from $M_{n,n}(k, k; R)$. For the sake of convenience it will be assumed that $A_{\alpha, \alpha'} = 0$ ($\alpha! = \infty$) whenever $\alpha \notin I_n$ or $\alpha' \notin I_{n'}$ (respect. $\alpha \notin I_n$).

Proposition 1. For the above defined product the following are true.

1. $A \odot B = B \odot A$.
2. $(A + B) \odot C = A \odot C + B \odot C$.
3. $(A \odot B) \odot C = A \odot (B \odot C)$
4. $(\lambda A) \odot B = \lambda(A \odot B)$ for any $\lambda \in F$
5. $A \odot B = 0$ if and only if $A = 0$ or $B = 0$.
6. $A(B \odot H) = (AB) \odot H$
7. $(E_k \odot V)A = A \odot V$

In future $A^{(m)}$ means the m th power of the matrix A with respect to the new product.

Proposition 2. If $h = (h_1, h_2, \dots, h_n) \in M(0, 1; F)$, $v = (v_1, v_2, \dots, v_n) \in M(1, 0; F)$, then

$$(h^{(m)})_{0, \alpha'} = \binom{m}{\alpha'} h^{\alpha'}, \quad (v^{(m)})_{\alpha, 0} = m! v^{\alpha}$$

, where h^{α} stands for $h_1^{\alpha_1} h_2^{\alpha_2} \dots h_n^{\alpha_n}$

In future let $\rho \geq 1$ be any fixed real number and ϱ stand for the real number for which $\frac{1}{\rho} + \frac{1}{\varrho} = 1$. We consider the following ρ -norm of elements $A \in M(p, p'; F)$:

Definition 2.

$$\|A\| = \|A\|_{\rho} = \left(\sum_{\alpha, \alpha'} \frac{|A_{\alpha, \alpha'}|^{\rho}}{\alpha! (p! p')^{\rho-1}} \right)^{1/\rho}$$

In the case of $\rho = \infty$ the ρ -norm is defined by

$$\|A\| = \|A\|_{\infty} = \frac{\sup_{\alpha, \alpha'} |A_{\alpha, \alpha'}|}{p! p'!}$$

Theorem 1. 1. If $A, B \in Mat(p, p'; F)$ and $\lambda \in F$ then

- a) $\|A\| = 0$ if and only if $A = 0$,
- b) $\|\lambda A\| = |\lambda| \|A\|$,
- c) $\|A + B\| \leq \|A\| + \|B\|$.

2. For any nonnegative integer numbers p, p', q and q' there is such a positive number $\lambda(p, p', q, q')$ that for any $A \in Mat(p, p'; F)$, $B \in Mat(q, q'; F)$ the following inequality is valid:

$$\lambda(p, p', q, q') \|A\| \|B\| \leq \|A \odot B\| \leq \|A\| \|B\|$$

Proof. Here is a proof of part 2. First let us show the inequality $\|A \odot B\| \leq \|A\| \|B\|$.

Due to the Hölder inequality for $A \odot B = C$ one has

$$\begin{aligned} |C_{\alpha, \alpha'}| &= \left| \sum_{\beta \ll \alpha, \beta' \ll \alpha'} \binom{\alpha}{\beta} A_{\beta, \beta'} B_{\alpha-\beta, \alpha'-\beta'} \right| \leq \alpha! \sum_{\beta \ll \alpha, \beta' \ll \alpha'} \frac{|A_{\beta, \beta'} B_{\alpha-\beta, \alpha'-\beta'}|}{(\beta! (\alpha-\beta)!)^{1/\rho}} \frac{1}{(\beta! (\alpha-\beta)!)^{1/\varrho}} \leq \\ &\alpha! \left(\sum_{\beta \ll \alpha, \beta' \ll \alpha'} \frac{|A_{\beta, \beta'}|^{\rho} |B_{\alpha-\beta, \alpha'-\beta'}|^{\rho}}{\beta! (\alpha-\beta)!} \right)^{1/\rho} \left(\sum_{\beta \ll \alpha, \beta' \ll \alpha'} \frac{1}{(\beta! (\alpha-\beta)!)^{1/\varrho}} \right)^{1/\varrho} = \\ &\alpha! \left(\sum_{\beta \ll \alpha, \beta' \ll \alpha'} \frac{|A_{\beta, \beta'}|^{\rho}}{\beta!} \frac{|B_{\alpha-\beta, \alpha'-\beta'}|^{\rho}}{(\alpha-\beta)!} \right)^{1/\rho} \left(\sum_{\beta \ll \alpha} \frac{1}{\beta! (\alpha-\beta)!} \sum_{\beta' \ll \alpha'} 1 \right)^{1/\varrho} \leq \\ &\alpha! \left(\sum_{\beta \ll \alpha, \beta' \ll \alpha'} \frac{|A_{\beta, \beta'}|^{\rho}}{\beta!} \frac{|B_{\alpha-\beta, \alpha'-\beta'}|^{\rho}}{(\alpha-\beta)!} \right)^{1/\rho} \binom{p+q}{p} \frac{1}{\alpha!} \binom{p'+q'}{p'}^{1/\varrho} \end{aligned}$$

as far as according to Proposition 1 one has $\sum_{\beta \ll \alpha} \frac{1}{\beta!(\alpha-\beta)!} = \left(\frac{p+q}{p} \right) \frac{1}{\alpha!}$ and $\sum_{\beta' \ll \alpha'} 1 \leq \left(\frac{p'+q'}{p'} \right)$. Therefore

$$\begin{aligned} \|C\| &= \left(\sum_{\alpha, \alpha'} \frac{|C_{\alpha, \alpha'}|^\rho}{\alpha!((p+q)!(p'+q')!)^{\rho-1}} \right)^{1/\rho} \leq \\ & \left(\sum_{\alpha, \alpha'} \frac{1}{\alpha!((p+q)!(p'+q')!)^{\rho-1}} (\alpha!)^\rho \sum_{\beta \ll \alpha, \beta' \ll \alpha'} \frac{|A_{\beta, \beta'}|^\rho}{\beta!} \frac{|B_{\alpha-\beta, \alpha'-\beta'}|^\rho}{(\alpha-\beta)!} \left(\frac{p+q}{p} \right) \frac{1}{\alpha!} \left(\frac{p'+q'}{p'} \right)^{\rho/\varrho} \right)^{1/\rho} = \\ & \left(\sum_{\beta, \beta'} \frac{|A_{\beta, \beta'}|^\rho}{\beta!(p!p')^{\rho-1}} \right)^{1/\rho} \left(\sum_{\gamma, \gamma'} \frac{|B_{\gamma, \gamma'}|^\rho}{\gamma!(q!q')^{\rho-1}} \right)^{1/\rho} = \|A\| \|B\| \end{aligned}$$

due to $\rho/\varrho = \rho - 1$

To show the inequality $\lambda(p, p', q, q') \|A\| \|B\| \leq \|A \odot B\|$ let us consider

$$X = \{(A, B) : A \in \text{Mat}(p, p'; F), \|A\| = 1, B \in \text{Mat}(q, q'; F), \|B\| = 1\}$$

, which is a compact set in the corresponding finite dimensional vector space, and the continuous map $(A, B) \mapsto A \odot B$. The image of X , with respect to this map, is a compact set which doesn't contain zero vector due to Proposition 1. Let $\lambda(p, p', q, q') > 0$ stand for the distance between zero vector and this image set with respect to the corresponding ρ -norm. So $\lambda(p, p', q, q') \leq \|A \odot B\|$ for any $(A, B) \in X$ and due to Proposition 1 one has

$$\lambda(p, p', q, q') \|A\| \|B\| \leq \|A \odot B\|$$

for any $A \in \text{Mat}(p, p'; F), B \in \text{Mat}(q, q'; F)$

Remark. It would be nice if one could offer an expression for

$$\lambda(p, p', q, q') = \inf_{\|A(p, p')\| = \|B(q, q')\| = 1} \|A \odot B\|_\rho$$

in terms of n, n', p, p', q, q' and ρ .

With respect to the ordinary product of matrices a result similar to $\|A \odot B\| \leq \|A\| \|B\|$ is not valid. But one can have the following result.

Proposition 3. The following inequality

$$\|A(p, q)B(q, q')\| \leq (q!)^{2-1/\rho} (q')^{2/\rho-1} \|A\| \|B\|_\varrho$$

is true.

Proof. Indeed due to the Hölder inequality one has

$$\begin{aligned} \|A(p, q)B(q, q')\|^\rho &= \sum_{\alpha, \alpha'} \frac{1}{\alpha!(p!q!)^{\rho-1}} \left| \sum_{\beta} A_{\alpha, \beta} B_{\beta, \alpha'} \right|^\rho \leq \sum_{\alpha, \alpha'} \frac{1}{\alpha!(p!q!)^{\rho-1}} \sum_{\beta} |A_{\alpha, \beta}|^\rho \left(\sum_{\gamma} |B_{\gamma, \alpha'}|^\varrho \right)^{\rho/\varrho} = \\ & \sum_{\alpha, \beta} \frac{|A_{\alpha, \beta}|^\rho}{\alpha!(p!q!)^{\rho-1}} \left(\sum_{\gamma, \alpha'} \frac{|B_{\gamma, \alpha'}|^\varrho}{\gamma!(q!q')^{\varrho-1}} \gamma! \right)^{\rho/\varrho} (q!)^\rho (q')^{2-\rho} \leq \|A\|^\rho \|B\|_\varrho^\rho (q!)^{2\rho-1} (q')^{2-\rho} \end{aligned}$$

as far as $\gamma! \leq q!$.

In particular case the following estimation is also true.

Proposition 4. For any nonnegative integer numbers m, k, q' and $h \in \text{Mat}_{n, n}(0, 1; F)$, $A \in \text{Mat}_{n, n'}(m+k, q'; F)$ the following inequality

$$\left\| \left(\frac{h^{(m)}}{m!} \odot E_k \right) A \right\| \leq \binom{m+k}{k} \|h\|_\varrho^m \|A\|$$

is valid.

Proof. Indeed

$$\begin{aligned} \|(\frac{h^{(m)}}{m!} \odot E_k)A\|^\rho &= \sum_{\alpha, \alpha'} \frac{1}{\alpha!(k!q'!)^{\rho-1}} |(\frac{h^{(m)}}{m!} \odot E_k)A_{\alpha, \alpha'}|^\rho = \sum_{\alpha, \alpha'} \frac{1}{\alpha!(k!q'!)^{\rho-1}} |\sum_{\beta} (\frac{h^{(m)}}{m!} \odot E_k)_{\alpha, \beta} A_{\beta, \alpha'}|^\rho = \\ &= \sum_{\alpha, \alpha'} \frac{1}{\alpha!(k!q'!)^{\rho-1}} |\sum_{\beta} \frac{h^{\beta-\alpha}}{(\beta-\alpha)!^{1/\rho}} \frac{A_{\beta, \alpha'}}{(\beta-\alpha)!^{1/\rho}}|^\rho \end{aligned}$$

as far as

$$(\frac{h^{(m)}}{m!} \odot E_k)_{\alpha, \beta} = \frac{h^{\beta-\alpha}}{(\beta-\alpha)!}$$

Due to the Hölder inequality

$$(\sum_{\beta} |\frac{h^{\beta-\alpha}}{(\beta-\alpha)!^{1/\rho}} \frac{A_{\beta, \alpha'}}{(\beta-\alpha)!^{1/\rho}}|)^\rho \leq \sum_{\beta} \frac{|A_{\beta, \alpha'}|^\rho}{(\beta-\alpha)!} (\sum_{\beta} \frac{|h^{\beta-\alpha}|}{(\beta-\alpha)!})^{\rho/\rho} = \sum_{\beta} \frac{|A_{\beta, \alpha'}|^\rho}{(\beta-\alpha)!} (\frac{\|h\|_\rho^{\rho}}{m!})^{\rho/\rho}$$

Therefore

$$\begin{aligned} \|(\frac{h^{(m)}}{m!} \odot E_k)A\|^\rho &\leq \sum_{\beta, \alpha'} \frac{|A_{\beta, \alpha'}|^\rho}{\beta!((m+k)!q'!)^{\rho-1}} \binom{m+k}{k}^{\rho-1} \sum_{\alpha} \binom{\beta}{\alpha} \|h\|_\rho^{m\rho} = \\ &= \sum_{\beta, \alpha'} \frac{|A_{\beta, \alpha'}|^\rho}{\beta!((m+k)!q'!)^{\rho-1}} \binom{m+k}{k}^\rho \|h\|_\rho^{m\rho} = \|A\|^\rho \binom{m+k}{k}^\rho \|h\|_\rho^{m\rho} \end{aligned}$$

Corollary. For any nonnegative integer number $m, q', A \in Mat_{n, n'}(m, q'; F)$ and $h^i \in Mat_{n, n}(0, 1; F), i = 1, 2, \dots, m$ the following inequality

$$\|\frac{h^1 \odot h^2 \odot \dots \odot h^m}{m!} A\| \leq \|h^1\|_\rho \|h^2\|_\rho \dots \|h^m\|_\rho \|A\|$$

is valid.

From now let us assume that x_1, x_2, \dots, x_n are variables over F, q' is a fixed nonnegative integer and $x = (x_1, x_2, \dots, x_n) \in M_{n, n}(0, 1; F[x])$.

Now we are going to consider an application of the new product to power series $\sum_{\alpha \in I_n} x^\alpha a_\alpha$, where $a_\alpha \in Mat(0, q'; F)$. To do it we represent the power series $\sum_{\alpha \in I_n} x^\alpha a_\alpha$ in the form $\sum_{m=0}^\infty \frac{x^{(m)}}{m!} A(m)$, where $A(m) \in M(m, q'; F)$.

It is well known that all ρ -norms define the same topology in F^n . Therefore one can speak about convergence of the above power series without refereing to any particular ρ -norm.

Definition 3. A power series $\sum_{m=0}^\infty \frac{x^{(m)}}{m!} A(m)$ is said to be absolute convergent at $h \in F^n$ if its each component is absolute convergent at $h \in F^n$ e.i. for each $\alpha' \in I_{n'}, |\alpha'| = q'$, the positive series $\sum_{\alpha \in I_n} |\frac{h^\alpha}{\alpha!} A(m)_{\alpha, \alpha'}|$ converges.

Due to the inequality

$$|\frac{h^{(m)}}{m!} A(m)| \leq \|h\|_\rho^m \|A(m)\|$$

(Proposition 4) the power series

$$\sum_{m=0}^\infty \frac{x^{(m)}}{m!} A(m) \tag{1}$$

absolutely converges whenever $\|x\|_\rho < R = \frac{1}{r}$, where $r = \overline{\lim}_{m \rightarrow \infty} \|A(m)\|_\rho^{\frac{1}{m}}$.

Theorem 2. Power series (1) is absolute convergent at $h \in F^n$ whenever $\|h\|_\rho < R$ and for any $R_1 > Rn^{\frac{\rho-1}{\rho}}$ there exists such $\bar{h} \in F^n$ that $\|\bar{h}\|_\rho = R_1$ and power series (1) is not absolute convergent at \bar{h}

Proof.

If $R = \infty$ there is nothing to prove. Assume that $1 \leq \rho < \infty, R < \infty, R_1 > Rn^{\frac{\rho-1}{\rho}}$ and for any $\bar{h} \in F^n$ for which $\|\bar{h}\|_\rho = R_1$ power series (1) is absolute convergent at \bar{h} . Due to convergence of the series

$\sum_{m=0}^{\infty} \sum_{|\alpha|=m} \left| \frac{\bar{h}^\alpha}{\alpha!} A(m)_{\alpha, \alpha'} \right|$ for big enough m one has

$$\left| \frac{\bar{h}^\alpha}{\alpha!} A(m)_{\alpha, \alpha'} \right| \geq \left| \frac{\bar{h}^\alpha}{\alpha!} A(m)_{\alpha, \alpha'} \right|^\rho \geq |\bar{h}^\alpha|^\rho \frac{|A(m)_{\alpha, \alpha'}|^\rho}{\alpha! (m! q'!)^{\rho-1}}$$

and therefore the series

$$\sum_{|\alpha'|=q'}^{\infty} \sum_{m=0}^{\infty} \sum_{|\alpha|=m} |\bar{h}^\alpha|^\rho \frac{|A(m)_{\alpha, \alpha'}|^\rho}{\alpha! (m! q'!)^{\rho-1}}$$

should converge whenever $\bar{h} \in F^n$ and $\|\bar{h}\|_\varrho = R_1$. Consider this series for $\bar{h} = (R_1 n^{\frac{-1}{\varrho}}, R_1 n^{\frac{-1}{\varrho}}, \dots, R_1 n^{\frac{-1}{\varrho}}) \in F^n$ for which $\|\bar{h}\|_\varrho = R_1$.

$$\sum_{|\alpha'|=q'}^{\infty} \sum_{m=0}^{\infty} \sum_{|\alpha|=m} |\bar{h}^\alpha|^\rho \frac{|A(m)_{\alpha, \alpha'}|^\rho}{\alpha! (m! q'!)^{\rho-1}} = \sum_{m=0}^{\infty} (R_1 n^{\frac{-1}{\varrho}})^{m\rho} \sum_{|\alpha|=m, |\alpha'|=q'} \frac{|A(m)_{\alpha, \alpha'}|^\rho}{\alpha! (m! q'!)^{\rho-1}} = \sum_{m=0}^{\infty} (R_1 n^{\frac{-1}{\varrho}})^{m\rho} \|A(m)\|^\rho$$

But the radius of convergence of the ordinary number series $\sum_{m=0}^{\infty} t^m \|A(m)\|^\rho$ equals R^ρ and $(R_1 n^{\frac{-1}{\varrho}})^\rho > R^\rho$ this contradiction indicates that the Theorem is true in this case.

Let us consider now the $\rho = \infty$ case. Assume that $R_1 > Rn$ and power series (1) is absolute convergent at any $\bar{h} \in F^n$ for which $\|\bar{h}\|_1 = R_1$. In particular for any $\alpha' \in I_{n'}$, $|\alpha'| = q'$ and $\bar{h} = (R_1 n^{\frac{-1}{\varrho}}, R_1 n^{\frac{-1}{\varrho}}, \dots, R_1 n^{\frac{-1}{\varrho}}) \in F^n$, for which $\|\bar{h}\|_1 = R_1$, the series

$$\sum_{m=0}^{\infty} \sum_{|\alpha|=m} \left| \frac{\bar{h}^\alpha}{\alpha!} A(m)_{\alpha, \alpha'} \right| = \sum_{m=0}^{\infty} \left(\frac{R_1}{n} \right)^m \sum_{|\alpha|=m} \left| \frac{A(m)_{\alpha, \alpha'}}{\alpha!} \right|$$

is convergent. In this case the series

$$\sum_{m=0}^{\infty} \left(\frac{R_1}{n} \right)^m \sum_{|\alpha|=m, |\alpha'|=q'} \left| \frac{A(m)_{\alpha, \alpha'}}{m! q'!} \right|$$

is convergent as well. Due to the equality $\sum_{|\alpha|=m} 1 = \binom{m+n-1}{n-1}$ the following inequality

$$\|A(m)\|_\infty \leq \sum_{|\alpha|=m, |\alpha'|=q'} \left| \frac{A(m)_{\alpha, \alpha'}}{m! q'!} \right| \leq \|A(m)\|_\infty \binom{m+n-1}{n-1} \binom{q'+n'-1}{n'-1}$$

is clear. It implies that the radius of convergence of the power series $\sum_{m=0}^{\infty} t^m \sum_{|\alpha|=m, |\alpha'|=q'} \left| \frac{A(m)_{\alpha, \alpha'}}{m! q'!} \right|$ is the same R . But in our case $\frac{R_1}{n} > R$ and $\sum_{m=0}^{\infty} \left(\frac{R_1}{n} \right)^m \sum_{|\alpha|=m, |\alpha'|=q'} \left| \frac{A(m)_{\alpha, \alpha'}}{m! q'!} \right|$ converges. This contradiction indicates that the Theorem is true in this case as well.

Due to this result if for power series (1) the radius of convergence R is zero then in any neighborhood of zero one can find a point where (1) is not absolute convergent. This theorem indicates also a privileged position of 1-norm among all ρ -norms as far as in this case $Rn^{\frac{\rho-1}{\rho}} = R$ and for $\rho > 1$ power series (1) has a solid layer

$$\{h \in F^n : R \leq \|h\|_\varrho \leq Rn^{\frac{\rho-1}{\rho}}\}$$

of indeterminacy. Investigation the behavior of power series (1) in this layer of indeterminacy could be interesting. It is hoped that the emergence of a layer of indeterminacy is not a defect of our approach.

Question 1. Let $\sum_\alpha a_\alpha x^\alpha$ be any series for which $\sum_{m=0}^{\infty} H_m(x)$, where $H_m(x) = \sum_{|\alpha|=m} a_\alpha x^\alpha$, is absolute convergent in some neighborhood of zero. Does it imply that the original series $\sum_\alpha a_\alpha x^\alpha$ is also absolute convergent in some neighborhood of zero as well?

Question 2. Consider power series (1), for each nonnegative m multilinear map

$$(F^n)^m \longrightarrow F^{n'} : (h^1, h^2, \dots, h^m) \longmapsto (h^1 \odot h^2 \odot \dots \odot h^m) A(m)$$

and its norm $\|A(m)\|$ ([2]) defined by

$$\|A(m)\| = \sup_{\|h^1\|_\varrho = \|h^2\|_\varrho = \dots = \|h^m\|_\varrho = 1} \|(h^1 \bigodot h^2 \bigodot \dots \bigodot h^m)A(m)\|_\rho$$

According to the above Corollary for each m the inequality $\|A(m)\| \leq \|A(m)\|_\rho$ is true. Is it true that

$$\overline{\lim}_{m \rightarrow \infty} \|A(m)\|^{\frac{1}{m}} = \overline{\lim}_{m \rightarrow \infty} \|A(m)\|_\rho^{\frac{1}{m}}$$

References

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